Distributed control of DC microgrids using primal-dual dynamics

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***Abstract*— In this paper, a novel distributed control strategy achieving *(feasible)* current sharing and voltage regulation in Direct Current (DC) microgrids is proposed. Firstly, the (convex) optimization problem is formulated, with the microgrid's steady state equations and/or desired objectives as feasible set. Secondly, we design a controller, the (unforced) dynamics of which represent the continuous time primal-dual dynamics of the considered optimization problem. Then, a passive interconnection between the physical plant and the controller is presented. Furthermore, global asymptotic convergence of the closed-loop system to the desired steady-state is proved and simulations successfully confirm the theoretical results.**

I. INTRODUCTION

The recent wide spread of renewable energy sources motivates the design and operation of Direct Current (DC) microgrids, which are interconnected clusters of Distributed Generation Units (DGUs), loads and energy storage systems interacting each other through distribution lines [1]. In order to guarantee a proper and safe functioning on the power network, voltage stabilization is the main goal to achieve in DC microgrids [2]. Additionally, to avoid the overstressing of a source, it is generally desired that the total demand is (fairly) shared among all the DGUs of the microgrid [3]. However, in order to permit the DGUs to share the generated current or power, voltage differences among the nodes of the microgrid are necessary. As a consequence, it is generally not possible to achieve the aforementioned objectives simultaneously.

In the literature, several control techniques have been proposed to regulate the voltages towards the corresponding desired value (see for instance [4] and the references therein). On the other hand, some works have proposed consensus-based control schemes achieving current/power sharing without regulating the voltage (see for instance [5] and the references therein). Differently from the above mentioned works, consensus protocols have been recently designed for achieveing both current sharing and a peculiar form of voltage regulation, where the average value of the voltages of the whole microgrid is controlled towards a desired setpoint (see for instance [3], [6]–[8] and the references therein). However, regulating *only* the average voltage may introduce, in some nodes of the microgrid, large voltage deviations from the corresponding nominal value, making this solution not always adequate in practical applications.

All these considerations motivated us to design a control scheme aiming to share, among the nodes of the microgrid, the largest possible amount of total demand in compliance with the permitted (safe) values of voltage deviations (a similar control problem is addressed in [9] by synthesizing a centralized symbolic controller). More precisely, in this paper, we consider a DC microgrid with buck converters and loads interconnected through resistive-inductive power lines. For the considered DC microgrid we propose a novel distributed control scheme that achieves *feasible* current sharing, keeping the (steady-state) voltage at each node within prescribed desired bounds. This is done by coupling fundamental concepts from convex optimization and systems theory, i.e., (continuous) primal-dual dynamics [10]–[12] and passivity [13]. In analogy with [14], the proposed design procedure involves the following key steps: (i) The (convex) optimization problem is formulated and the corresponding feasible set is defined by the microgrid's steady state conditions and/or desired objectives. (ii) A dynamic controller, the (unforced) dynamics of which represent the primal-dual dynamics of the considered optimization problem is designed. (iii) After showing the passivity property of the microgrid and controller, a power-conserving interconnection between the microgrid and the controller is established and the (global) asymptotic convergence of the closed-loop system trajectories to the desired equilibrium point is proved.

II. MICROGRID'S MODEL AND PASSIVITY PROPERTY

In this paper we consider a typical DC microgrid with n Distributed Generation Units (DGUs) connected to each other through m resistive-inductive (RL) lines (see [15, Fig. 1] for a schematic electrical diagram of the considered DC network and Table I for the description of the used symbols). Each DGU includes a DC-DC buck converter supplying a "ZI" (constant impedance, constant current) load*. The DC load is connected to the so-called Point of Common Coupling (PCC). The overall DC microgrid is represented by a connected and undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where the nodes, $\mathcal{V} = \{1, ..., n\}$, represent the DGUs and the edges, $\mathcal{E} = \{1, ..., m\}$, represent the lines interconnecting the DGUs. The microgrid topology is described by its corresponding incidence matrix $\mathcal{B} \in \mathbb{R}^{n \times m}$. The ends of edge $k \in \mathcal{E}$ are arbitrarily labeled with a $+$ and a $-$, and the entries of $\mathcal{B}$ are given by $\mathcal{B}_{ik} = +1$ if i is the positive

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*In the presence of also constant power-loads, the results in this paper hold locally.

TABLE I

DESCRIPTION OF THE USED SYMBOLS

	States and input		Parameters
I_{ti}	Generated current	R_{ti}, L_{ti}	Filter resistance, inductance
V_i	Load voltage	C_{ti}	Shunt capacitor
I_k	Line current	R_k, L_k	Line resistance, inductance
u_i	Control input	Z_{li}, I_{li}	Load impedance, current

end of k, $\mathcal{B}_{ik} = -1$ if i is the negative end of k, and $\mathcal{B}_{ik} = 0$ otherwise. Consequently, the overall dynamical system describing the microgrid behaviour can be written compactly for all DGUs $i \in \mathcal{V}$ as

$$\begin{aligned} L_t\dot{I}_t &= -V - R_tI_t + u \\ L\dot{I} &= -RI - \mathcal{B}^\top V \\ C_t\dot{V} &= I_t + \mathcal{B}I - Z_l^{-1}V - I_l, \end{aligned} \tag{1}$$

where $I_t, V, u : \mathbb{R} \to \mathbb{R}^n$, $I : \mathbb{R} \to \mathbb{R}^m$ and $I_l \in \mathbb{R}^n$. Moreover, $C_t, L_t, R_t, Z_l \in \mathbb{R}^{n\times n}$ and $R, L \in \mathbb{R}^{m\times m}$ are positive definite diagonal matrices, e.g., $C_t = \mathrm{diag}(C_{t1}, \dots, C_{tn})$. Furthermore, let $x = [I_t^\top, I^\top, V^\top]^\top$ denote the state of system (1). Then, for a given constant input $\overline{u}$, the corresponding steady state solution $\overline{x} = [\overline{I}_t^\top, \overline{I}^\top, \overline{V}^\top]^\top$ to system (1) satisfies

$$\overline{V} = -R_t\overline{I}_t + \overline{u} \tag{2a}$$
$$\overline{I} = -R^{-1}\mathcal{B}^\top\overline{V} \tag{2b}$$
$$\overline{I}_t = -\mathcal{B}\overline{I} + Z_l^{-1}\overline{V} + I_l. \tag{2c}$$

Before establishing the passivity property of system (1), we first define the set of all feasible forced equilibria of (1) as follows

$$\mathcal{E}_p := \{(\overline{x}, \overline{u}) \in \mathbb{R}^{2n+m} \times \mathbb{R}^n | (\overline{x}, \overline{u}) \text{ satisfies (2)}\}. \tag{3}$$

Remark 1: **(Unique steady state solution)**. Given $\overline{u}$, there exists a unique $(\overline{x}, \overline{u}) \in \mathcal{E}_p$ satisfying:

$$\begin{aligned} \overline{V} &= (\mathbb{I}_n + R_tG)^{-1}(\overline{u} - R_tI_l) \\ \overline{I}_t &= G(\mathbb{I}_n + R_tG)^{-1}(\overline{u} - R_tI_l) + I_l \\ \overline{I} &= -R^{-1}\mathcal{B}^\top(\mathbb{I}_n + R_tG)^{-1}(\overline{u} - R_tI_l), \end{aligned}$$

with $G := \mathcal{B}R^{-1}\mathcal{B}^\top + Z_l^{-1}$.

Now, the following result can be proved [16].

Proposition 1: **(Property of** (1)**)**. Let $u_d : \mathbb{R} \to \mathbb{R}^n$. The following statements hold:

(a) System (1) together with $\dot{u} = u_d$ is passive with respect to the supply rate $u_d^\top \dot{I}_t$ and storage function†

$$S_p(I_t, I, V, u) = \frac{1}{2}\|\dot{I}_t\|^2_{L_t} + \frac{1}{2}\|\dot{I}\|^2_L + \frac{1}{2}\|\dot{V}\|^2_{C_t}. \tag{4}$$

(b) Let $u_d = 0$. System (1) converges to the equilibrium point $(\overline{x}, \overline{u}) \in \mathcal{E}_p$.

Remark 2: In Proposition 1, we consider the extended dynamics of the microgrid, i.e., system (1) together with $\dot{u} = u_d$. For the extended system, the states and input are I_t, I, V, u and u_d, respectively. This motivates us to consider a storage function (4) depending on states I_t, I, V and input u of the plant (see [18], [19] for further information on passivity properties of extended dynamics of general nonlinear systems).

†As noticed in [17, Remark 2], the storage function S_p in (4) depends on I_t, I, V and u. This is evident from replacing $\dot{I}_t, \dot{I}$ and $\dot{V}$ by the corresponding dynamics in (1).

III. PROBLEM FORMULATION

First, we notice that (2) can be expressed as

$$\overline{V} = -R_t\overline{I}_t + \overline{u}, \tag{5a}$$
$$\overline{I}_t = G\overline{V} + I_l. \tag{5b}$$

Then, let us define the set‡

$$\overline{\mathcal{E}}_p := \{(\overline{u}, \overline{I}_t, \overline{V}) \in \mathbb{R}^{3n} | (\overline{u}, \overline{I}_t, \overline{V}) \text{ satisfies (5)}\}. \tag{6}$$

Before formulating the control objective concerning the PCC voltages, we assume that for every DGU $i \in \mathcal{V}$, there exists a nominal reference voltage $V_i^\star$:

Assumption 1: **(Nominal voltages)**. There exists a reference voltage $V_i^\star > 0$ at the PCC, for all $i \in \mathcal{V}$.

Generally, in order to achieve an efficient demand and supply matching, so avoiding the overstressing of a source, it is desirable that the total load demand of the microgrid is shared among all the DGUs proportionally to the generation capacity of their corresponding energy sources (fair current sharing). This desire is equivalent to achieve $w_i\overline{I}_{ti} = w_j\overline{I}_{tj}$ for all $i, j \in \mathcal{V}$, where a relatively large value of $w_i \in \mathbb{R}_+$ corresponds to a relatively small generation capacity of DGU i. We call this desire "*ideal* current sharing" and, in analogy with [6], [7] and [8], can be expressed as follows:

$$\lim_{t\to\infty} I_t(t) = \overline{I}_t = W^{-1}1 i_t^\star, \tag{7}$$

with $W = \mathrm{diag}(w_1, \dots, w_n)$, $w_i \in \mathbb{R}_+$, for all $i \in \mathcal{V}$ and $i_t^\star \in R_+$. More precisely, (2c) or, equivalently, (5b) implies $i_t^\star = 1^\top(Z_l^{-1}\overline{V} + I_l)/(1^\top W^{-1}1)$. Moreover, we notice that to achieve current sharing, each DGU needs to share with its neighbours information on the generated current. Then, let $\mathcal{L} = \mathcal{B}\Gamma\mathcal{B}^\top$ denote the (weighted) Laplacian matrix associated to the communication network, where $\Gamma \in \mathbb{R}^{m\times m}$ is a positive definite diagonal matrix describing the weights on the edges.

However, achieving *ideal* current sharing, prescribes the value of the required differences in voltages among the nodes of the network. As a consequence, it is generally not possible to control the voltage at each node towards the corresponding desired value. For this reason, the voltage requirements are generally relaxed and, as an alternative, several control approaches in the literature propose to regulate the (weighted) "*average* voltage" across the whole microgrid towards a global voltage set point [3], [6]–[8], where the sources with the largest generation capacity determine the grid voltage, i.e.,

$$\lim_{t\to\infty} 1^\top W^{-1}V(t) = 1^\top W^{-1}\overline{V} = 1^\top W^{-1}V^\star. \tag{8}$$

‡Note that $(\overline{u}, \overline{I}_t, \overline{V}) \in \overline{\mathcal{E}}_p \iff (\overline{I}_t, -R^{-1}\mathcal{B}^\top\overline{V}, \overline{V}, \overline{u}) \in \mathcal{E}_p$.

We now observe that achieving *ideal* current sharing (7) and average voltage regulation (8) is equivalent to satisfy the following equalities:

$$\mathcal{B}^\top W \overline{I}_t = 0 \tag{9a}$$

$$\overline{V} = V^\star + W\mathcal{B}\overline{\eta}, \tag{9b}$$

with $\overline{\eta} \in \mathbb{R}^m$. This motivated us to adopt an approach similar to [14], [20]. More precisely, we propose the following (equality constrained) optimization problem with the steady-state equations (5) and desired objectives (9) as constraints[§]

$$\begin{aligned} &\underset{(\hat{u},\hat{I}_t,\hat{V})\in\overline{\mathcal{E}}_p,\eta\in\mathbb{R}^m}{\text{minimize}} \quad \mathcal{F}(\hat{u},\hat{I}_t,\hat{V},\eta) \\ &\text{subject to: } \ \mathcal{B}^\top W\hat{I}_t = 0, \ \hat{V} - V^* - W\mathcal{B}\eta = 0, \end{aligned} \tag{10}$$

where $\mathcal{F}(\hat{u},\hat{I}_t,\hat{V},\eta) \in C^1$ is a convex function in its arguments[¶]. However, we note that even achieving *ideal* current sharing preserving the average voltage of the microgrid may not always be desired, as it may introduce, in some nodes of the microgrid, large voltage deviations from the corresponding nominal value. Consider for instance a DC microgrid with 2 DGUs interconnected through a pure resistive line, the value of which is relatively large (e.g., because the DGUs are physically distant). Moreover, assume that the load demand in one of the DGUs is much higher then the other. Then, in order to achieve *ideal* current sharing, the DGUs need to share a relatively large current through the interconnecting line, implying a relatively large voltage deviation (with respect to the nominal value) at the corresponding PCCs. Consequently, a steady-state solution satisfying (10) may be not *feasible* in practical applications. To address this issue, first we modify (10) as follows:

Objective 1: **(*Feasible* current sharing)**.

$$\underset{(\hat{u},\hat{I}_t,\hat{V})\in\overline{\mathcal{E}}_p}{\text{minimize}} \quad \mathcal{F}(\hat{u},\hat{I}_t,\hat{V}), \tag{11}$$

with

$$\mathcal{F} := \frac{\alpha}{2}\|\hat{u}\|^2 + \frac{\beta}{2}\|\mathcal{B}^\top W\hat{I}_t\|^2_\Gamma + \frac{\gamma}{2}\|\hat{V} - V^*\|^2, \tag{12}$$

where $\alpha, \beta, \gamma \in \mathbb{R}_+$ are design parameters.

Before providing the rationale of Objective 1, we introduce a second control objective concerning the desired steady-state value of the voltage. More precisely, in order to guarantee a proper functioning of the connected loads, it is generally required that the voltages remain within prescribed limits (see for instance [21] and the references therein). Then, differently from [3], [6]–[8], we consider in this paper the following steady-state constraint:

Objective 2: **(Voltage requirement)**.

$$V_{mi} \le \lim_{t\to\infty} V_i(t) \le V_{Mi}, \tag{13}$$

[§]Note that we assume to know the load. This is a reasonable assumption if controllable loads are considered. However, extending the results of this paper towards controllable loads is left as an interesting future endeavor.

[¶]Let x be the state of the physical plant, $\hat{x}$ denotes the corresponding optimization variable. Moreover, $\overline{x}$ and $\hat{x}^*$ denote the values of x and $\hat{x}$ at steady-state, respectively.

where V_{mi} and V_{Mi} denote the minimum and maximum permitted voltage value at the PCC of DGU i, for all $i \in \mathcal{V}$.

Remark 3: **(Rationale behind the control objectives)**. The quadratic function (12) comprises three different terms concerning (i) the control action, (ii) the current sharing and (iii) the voltage deviation from the corresponding nominal value. As a consequence, a solution to Objective 1 satisfying also Objective 2, generally does not guarantee the achievement of *ideal* current sharing (7). This is indeed equivalent to adopt a priority scale, where, in order to ensure a proper functioning of the microgrid, the voltage requirement (13) has a priority higher than current sharing. In other words, we are interested in a *feasible* solution that permits to share among the nodes of the microgrid the largest possible amount of total demand in compliance with the voltage requirement (13). Moreover, we notice that, achieving Objective 1 and Objective 2 does not always exclude the achievement of *ideal* current sharing.

IV. Distributed Primal-dual controller

In this section we present a basic primal-dual dynamic controller that achieve Objective 2 and (approximately) Objective 1. For the sake of exposition, in this subsection we provisionally exclude the inequality constraints (13), i.e., Objective 2, from the feasibility set. However, we include them in Subsection IV-A.

Consider the equality constrained optimization problem (11). Let $\lambda_a, \lambda_b \in \mathbb{R}^n$ denote the Lagrange multipliers corresponding to the constraints $\hat{V} + R_t\hat{I}_t - \hat{u} = 0$ and $\hat{I}_t - G\hat{V} - I_l = 0$, respectively. Moreover, let $\lambda = [\lambda_a^\top, \lambda_b^\top]^\top$ and $x_c = [\hat{u}^\top, \hat{I}_t^\top, \hat{V}^\top, \lambda^\top]^\top$. The Lagrangian function corresponding to the optimization problem (11) is

$$\mathcal{L}(x_c) := \mathcal{F} + \lambda_b^\top(\hat{I}_t - G\hat{V} - I_l) + \lambda_a^\top(\hat{V} + R_t\hat{I}_t - \hat{u}). \tag{14}$$

Consequently, the first order optimality conditions are given by Karush-Kuhn-Tucker (KKT) conditions, i.e.,

$$\alpha\hat{u}^* - \lambda_a^* = 0 \tag{15a}$$

$$\beta\mathcal{L}^w\hat{I}_t^* + \lambda_b^* + R_t\lambda_a^* = 0 \tag{15b}$$

$$\lambda_a^* - G\lambda_b^* + \gamma\left(\hat{V}^* - V^*\right) = 0 \tag{15c}$$

$$\hat{V}^* + R_t\hat{I}_t^* - \hat{u}^* = 0 \tag{15d}$$

$$\hat{I}_t^* - G\hat{V}^* - I_l = 0, \tag{15e}$$

where $\mathcal{L}_w = W\mathcal{L}W$. Moreover, we notice that the optimization problem is convex and the feasibility set $\overline{\mathcal{E}}_p$ is nonempty. As a consequence, the optimization problem satisfies Slater's condition and, therefore, strong duality holds [22]. Hence, $\hat{I}_t^*, \hat{V}^*, \hat{u}^*$ are optimal if and only if there exist λ_a^*, λ_b^* satisfying (15).

Now, consider the following dynamic controller, designed using the primal-dual dynamics of the optimization prob-

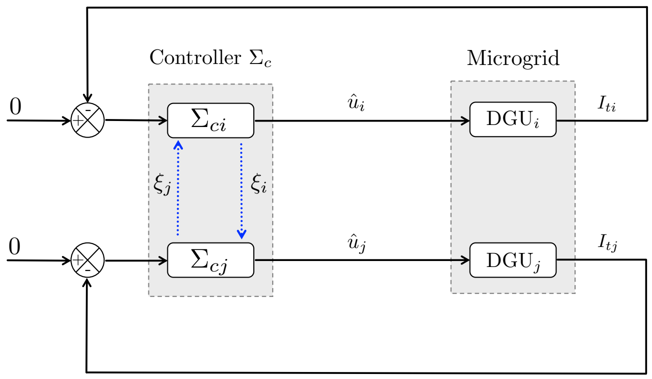


Fig. 1. Scheme of the proposed distributed primal-dual controller, where $j \in \mathcal{N}_i$ and $\xi_i = (\hat{I}_{ti}, \hat{V}_i, \lambda_{bi}), i \in \mathcal{V}$.

lem (11)

$$\begin{aligned}
-\tau_u \dot{\hat{u}} &= \alpha \hat{u} - \lambda_a - \nu_1 && \text{(16a)}\\
-\tau_I \dot{\hat{I}}_t &= \beta \mathcal{L}^w \hat{I}_t + \lambda_b + R_t \lambda_a && \text{(16b)}\\
-\tau_V \dot{\hat{V}} &= \lambda_a - G\lambda_b + \gamma\left(\hat{V} - V^*\right) - \nu_2 && \text{(16c)}\\
\tau_a \dot{\lambda}_a &= \hat{V} + R_t \hat{I}_t - \hat{u} && \text{(16d)}\\
\tau_b \dot{\lambda}_b &= \hat{I}_t - G\hat{V} - I_l, && \text{(16e)}
\end{aligned}$$

where $\tau_u, \tau_I, \tau_V, \tau_a, \tau_b > 0$ are design parameters. Let $\nu = [\nu_1^\top, \nu_2^\top]^\top$, ν_1, $\nu_2 : \mathbb{R} \to \mathbb{R}^n$ denote the controller input ports, which will be used later to interconnect the controller with the plant and to include the nodal constraints (see Figure 1). We now define the forced equilibria set of system (16), i.e.,

$$\mathcal{E}_{pd} := \{x_c, \nu_1, \nu_2 | \dot{x}_c = 0, \nu_1 = \nu_1^*, \nu_2 = \nu_2^*\}, \tag{17}$$

where $\nu_1^*, \nu_2^* \in \mathbb{R}^n$.

Remark 4: **(Unique steady state solution)** Note that the steady-state conditions of (16) represent the KKT conditions of optimization problem (11) with $\mathcal{F}$ replaced by $\mathcal{F}_1 := \mathcal{F} + \hat{u}^\top \nu_1^* + \hat{V}^\top \nu_2^*$. Since, the feasibility set $\overline{\mathcal{E}}_p$ of the optimization problem (11) is nonempty, $\mathcal{E}_{pd}$ is also nonempty. Also note that, given a pair of ν_1^*, ν_2^*, there exists a unique x_c such that $(x_c, \nu_1^*, \nu_2^*) \in \mathcal{E}_{pd}$.

Proposition 2: **(Property of** (16)**)** Let $\nu_d : \mathbb{R} \to \mathbb{R}^{2n}$. Then the following statements hold:

(a) The primal-dual controller (16), together with $\dot{\nu} = \nu_d$, is passive with port variables ν_d, $y_c := [\dot{\hat{u}}^\top, \dot{\hat{V}}^\top]^\top$ and storage function $S_c(z, \nu) = (1/2)\|\dot{x}_c\|_\tau^2$.

(b) Further for $\nu_d = 0$, then the unforced dynamics converge asymptotically to a point in $\mathcal{E}_{pd}$.

The proof (that we omit due to space limitation) follows from computing the Lie derivative of $S_c(z, \nu)$ along the vector fields of (16).

The passive dynamic controller (16) is now interconnected to the microgrid (1) by choosing $u = \hat{u}$ and $\nu_1 = -I_t$ (see Figure 1). Consequently, we obtain the following closed loop system

$$\begin{aligned}
-L_t \dot{I}_t &= V + R_t I_t - \hat{u}\\
-L\dot{I} &= RI + \mathcal{B}^\top V\\
C_t \dot{V} &= I_t + \mathcal{B}I - Z_l^{-1} V - I_l\\
-\tau_u \dot{\hat{u}} &= \alpha \hat{u} - \lambda_a + I_t\\
-\tau_I \dot{\hat{I}}_t &= \beta \mathcal{L}^w \hat{I}_t + \lambda_b + R_t \lambda_a\\
-\tau_V \dot{\hat{V}} &= \lambda_a - G\lambda_b + \gamma(\hat{V} - V^*) - \nu_2\\
\tau_a \dot{\lambda}_a &= \hat{V} + R_t \hat{I}_t - \hat{u}\\
\tau_b \dot{\lambda}_b &= \hat{I}_t - G\hat{V} - I_l.
\end{aligned} \tag{18}$$

The set of all feasible operating points of (18) is defined as

$$\mathcal{E}_1 := \{(x, x_c, \nu_2) \,|\, (x, \hat{u}) \in \mathcal{E}_p, (x_c, -I_t, \nu_2) \in \mathcal{E}_{pd}\}. \tag{19}$$

We have the following basic result.

Proposition 3: **(Preliminary result)** Assume that $\mathcal{E}_1$ is nonempty. Let $\dot{\nu}_2 = \nu_{2d}, \nu_{2d} : \mathbb{R} \to \mathbb{R}^n$. Consider the closed-loop system (18). Then the following statements hold:

(a) The interconnected system is passive with storage function $S_1 = S_p + S_c$, and port-variables $\dot{\hat{V}}$ with ν_{2d}.

(b) Given $\nu_{2d} = 0$, the closed-loop system asymptotically stabilizes to an operating point in $\mathcal{E}_1$.

Consider the operating points $(\overline{I}_t, \overline{I}, \overline{V}, \overline{u}) \in \mathcal{E}_p$. From Remark 1, we know that $\overline{u}$ completely defines the operating point $(\overline{I}_t, \overline{I}, \overline{V}, \overline{u}) \in \mathcal{E}_p$. If one considers $(\overline{u}, \hat{I}_t^*, \hat{V}^*, \lambda_a^*, \lambda_b^*, -\overline{I}_t, 0) \in \mathcal{E}_{pd}$, then from Proposition 3 we have that $(\overline{I}_t, \overline{I}, \overline{V}, \overline{u}, \hat{I}_t^*, \hat{V}^*, \lambda_a^*, \lambda_b^*, -\overline{I}_t, 0) \in \mathcal{E}_1$ is an asymptotically stable equilibrium of the closed-loop system (18). This implies that $\hat{V}^* + R_t \hat{I}_t^* - \overline{u} = 0$ and $\hat{I}_t^* - G\hat{V}^* - I_l = 0$. Using $G = \mathcal{B}R^{-1}\mathcal{B}^\top + Z_l^{-1}$, this simplifies to $\hat{V}^* = (\mathbb{I}_n + R_t Z_l^{-1})^{-1}(\overline{u} - R_t I_l)$ and $\hat{I}_t^* = -G(\mathbb{I}_n + R_t Z_l^{-1})^{-1}(\overline{u} - R_t I_l)$. Again from Remark 1, we obtain $\overline{V} = \hat{V}^*$, $\overline{I}_t = \hat{I}_t^*$, and $\overline{I} = -R^{-1}\mathcal{B}^\top \hat{V}^*$.

A. Including nodal constraints

We now extend the results discussed in the previous subsection to include nodal constraints on the voltages (see Objective 2). In order to achieve this, we consider the following optimization problem

$$\begin{aligned}
&\underset{(\hat{u}, \hat{I}_t, \hat{V}) \in \overline{\mathcal{E}}_p}{\text{minimize}} \quad \mathcal{F}(\hat{u}, \hat{I}_t, \hat{V})\\
&\text{subject to}\\
&g(\hat{V}) = \begin{bmatrix} g_m(\hat{V}) \\ g_M(\hat{V}) \end{bmatrix} = \begin{bmatrix} V_m - \hat{V} \\ \hat{V} - V_M \end{bmatrix} \leq 0,
\end{aligned} \tag{20}$$

where V_m, $V_M \in \mathbb{R}^n$ are chosen from practical considerations. Therefore, we assume that there exist at least one $\hat{V}$ satisfying the inequality constraints strictly i.e., $g(\hat{V}) < 0$ and the steady-state equations (2). As a consequence, (20) satisfies Slater's condition. This further implies that, $(\hat{I}_t^*, \hat{V}, \hat{u}^*)$ is the optimal solution of (20), if and only if there exist $\lambda_a^*, \lambda_b^* \in \mathbb{R}^n$, $\mu_m^*, \mu_M^* \in \mathbb{R}^n_{\geq 0}$ that satisfy the following KKT conditions

$$\alpha\hat{u}^* - \lambda_a^* = 0 \quad (21a)$$
$$\beta\mathcal{L}^w\hat{I}_t^* + \lambda_b^* + R_t\lambda_a^* = 0 \quad (21b)$$
$$\lambda_a^* - G\lambda_b^* + \gamma(\hat{V}^* - V^*) + \mu_h^* - \mu_l^* = 0 \quad (21c)$$
$$\hat{V}^* + R_t\hat{I}_t^* - \hat{u}^* = 0 \quad (21d)$$
$$\hat{I}_t^* - G\hat{V}^* - I_l = 0 \quad (21e)$$
$$V_m - \hat{V} \leq 0, \quad \mu_m^* \geq 0, \quad \mu_m^{*\top}(V_m - \hat{V}^*) = 0 \quad (21f)$$
$$\hat{V} - V_M \leq 0, \quad \mu_M^* \geq 0, \quad \mu_M^{*\top}(\hat{V}^* - V_M) = 0. \quad (21g)$$

Denote $\mu = [\mu_m^\top, \mu_M^\top]^\top$. For $i \in \{1 \cdots 2n\}$, let $\mu_i,\ g_i(\cdot)$ denote the i^{th} element of $\mu,\ g(\cdot)$. We define the primal-dual dynamics corresponding to the inequality constraint $g_i(\omega_i) \leq 0$ as the following subsystem [10], [23]

$$\tau_{\mu_i}\dot{\mu}_i = (g_i(\omega_i))^+_{\mu_i} := \begin{cases} g_i(\omega_i) & \text{if } \mu_i > 0 \\ \max\{0, g_i(\omega_i)\} & \text{if } \mu_i = 0 \end{cases}, \quad (22)$$

where $\tau_{\mu_i} > 0$, and $\mu_i,\ \omega_i : \mathbb{R} \to \mathbb{R}$ denote the state and input of the subsystem (22). The above dynamical system has the following properties:

(i) When $g_i(\omega_i) < 0,\ \mu_i = 0$ then $(g_i(\omega_i))^+_{\mu_i}$ switches from $g_i(\omega_i)$ to 0, which also ensures non-negativity of $\mu_i(t)$. Hence, if the initial condition $\mu_i(0) \geq 0$, then $\mu(t) \geq 0,\ \forall t \geq 0$.

(ii) For a constant input $\omega_i = \omega_i^*$, the equilibrium set of (22) is characterized by (μ_i^*, ω_i^*) satisfying
$$\mathcal{E}_{\mu_i} = \{(\mu_i^*, \omega_i^*) | g_i(\omega_i^*) \leq 0, \mu_i^* \geq 0, \mu_i^*\,(g_i(\omega_i^*)) = 0\}. \quad (23)$$

(iii) The overall dynamics of $2n$ inequality constraints are compactly written as
$$\tau_\mu\dot{\mu} = \hat{g}(\omega, \mu) \quad (24)$$
where $\tau_\mu = \text{diag}\{\tau_{\mu_1}, \cdots, \tau_{\mu_{2n}}\}$, $(g_i(\omega_i))^+_{\mu_i}$ is the ith component of $\hat{g}(\omega, \mu)$.

System (24) has the following passivity property [24]–[26].

Proposition 4: **(Property of** (24)**)** Let $\mathcal{P}$ denotes the power set of $\{1 \cdots 2n\}$, define $\sigma : [0, \infty) \to \mathcal{P}$ as $\sigma(t) = \{i | g_i(\omega_i) < 0 \text{ and } \mu_i = 0, \forall i \in \{1 \cdots 2n\}\}$. Consider system (24) with $\dot{\omega} = w_d$, $w_d : \mathbb{R} \to \mathbb{R}^n$. Then the following hold.

(a) System (24) is passive with storage function $S_\sigma(\mu) = (1/2)\sum_{i\notin\sigma} \dot{\mu}_i^2\tau_{\mu_i}$ and supply-rate $w_d^\top \dot{y}_\mu$, where $y_{\mu_i} = \mu_i \frac{\partial g_i}{\partial \omega_i}$.

(b) Let $w_d = 0$, and consider a constant $w^* \in \mathbb{R}^n$. $\forall i \in \{1 \cdots 2n\}$, if there exist a $(\mu_i^*, \omega_i^*) \in \mathcal{E}_{\mu_i}$ then (μ, ω^*) converges to a point in $\cup_{i\in\{1\cdots 2n\}}\mathcal{E}_{\mu_i}$.

The passive system (24) is now interconnected to (18) using the following interconnection constraints

$$\begin{aligned} \omega_i &= \hat{V}_i, \\ \omega_{n+i} &= \hat{V}_i, \\ \nu_{2_i} &= -(\mu_i - \mu_{n+i}), \end{aligned} \quad (25)$$

where $\omega = (\omega_1, \cdots, \omega_n)$, $\hat{V} = (\hat{V}_1, \cdots, \hat{V}_n)$, $\nu_2 = (\nu_{2_1}, \cdots, \nu_{2_n})$, $\mu_m = (\mu_1, \cdots, \mu_n)$ and $\mu_M = (\mu_{n+1}, \cdots, \mu_{2n})$. This gives the following closed loop system

$$\begin{aligned} -L_t\dot{I}_t &= V + R_tI_t - \hat{u} \\ -L\dot{I} &= RI + \mathcal{B}^\top V \\ C_t\dot{V} &= I_t + \mathcal{B}I - Z_l^{-1}V - I_l \\ -\tau_u\dot{\hat{u}} &= \alpha\hat{u} - \lambda_a - \hat{I}_t \\ -\tau_I\dot{\hat{I}}_t &= \beta\mathcal{L}^w\hat{I}_t + \lambda_b + R_t\lambda_a \\ -\tau_V\dot{\hat{V}} &= \lambda_a - G\lambda_b + \gamma(\hat{V} - V^*) + \mu_M - \mu_m \\ \tau_a\dot{\lambda}_a &= \hat{V} + R_t\hat{I}_t - \hat{u} \\ \tau_b\dot{\lambda}_b &= \hat{I}_t - G\hat{V} - I_l \\ \tau_{\mu_m}\dot{\mu}_m &= (V_m - \hat{V})^+_{\mu_m} \\ \tau_{\mu_M}\dot{\mu}_M &= (\hat{V} - V_M)^+_{\mu_M}, \end{aligned} \quad (26)$$

where $\tau_{\mu_m} = \text{diag}\{\tau_{\mu_1}, \cdots, \tau_{\mu_n}\}$, and $\tau_{\mu_M} = \text{diag}\{\tau_{\mu_{n+1}}, \cdots, \tau_{\mu_{2n}}\}$. We notice that the equilibrium points of (26) are

$$\mathcal{E} := \Big\{(x, x_c, \mu) \,|(x, x_c, \mu_M - \mu_m) \in \mathcal{E}_1,\ (\mu_i, \hat{V}_i) \in \mathcal{E}_{\mu_i}, \forall i \in \{1 \cdots 2n\}\Big\} \quad (27)$$

Now, we present the final result of the paper.

Proposition 5: **(Main result)** Assume that $\mathcal{E}$ is nonempty. System (26) asymptotically converges to a point in $\mathcal{E}$.

V. Simulation results

In this section, we validate in simulation the proposed control strategy. We consider a DC microgrid with four interconnected DGUs (see [15, Fig. 2]). The minimum and maximum permitted voltage values at PCC of all DGUs are set equal to 379 V and 381 V, respectively (see [15, Tables II, III] and Table II, for other system parameters). The controller parameters are $\tau_{(\cdot)} = 10^{-3}$, and $\alpha = \beta = \gamma = 1$. For the sake of illustration we consider the weighting matrix W to be the identity matrix. Figures 2(a) and 2(b) show in the first three seconds that the generated currents achieve *ideal* current sharing and voltages are within the permitted values. At the time instant $t = 3$ s, the load demand of node 3 is increased. Then, we observe that the generated voltages converge to the corresponding upper and lower limits to allow the largest possible sharing of load demand among the DGUs.

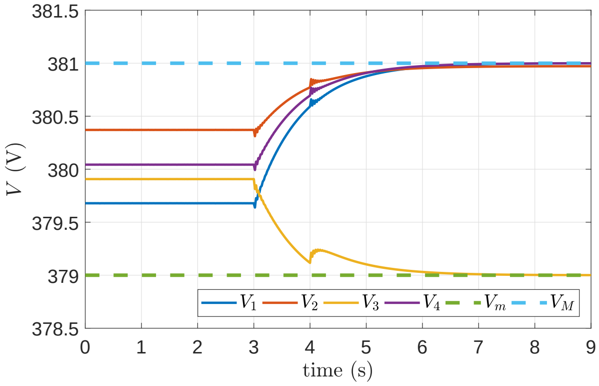
TABLE II

Microgrid Parameters

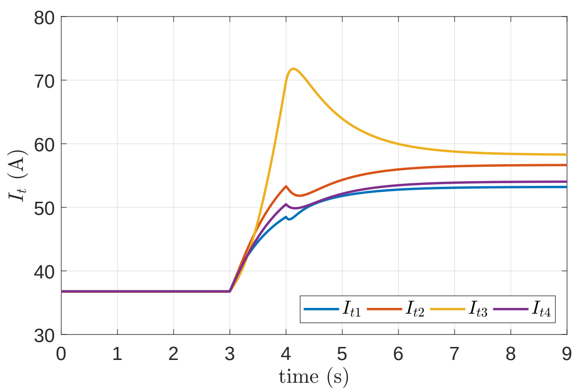
DGU		1	2	3	4
$Z_{li}(0)$	(Ω)	16.7	50.0	16.7	20.0
ΔG_l	(Ω^{-1})	0.03	0.02	-0.03	0.01
ΔI_{li}	(A)	0.0	0.0	75.0	0.0

VI. Conclusions and Future work

This paper explores the idea of using continuous time primal-dual dynamics for controlling DC microgrids in order to achieve current sharing and voltage regulation. We show that the proposed control scheme asymptotically stabilizes



(a) PCC voltages



(b) Generated currents

Fig. 2. Time evolution of the voltage at the PCC of each DGU (a); ; current generated by each DGU together (b).

the plant to a *feasible* operating point achieving the desired goals. Interesting future research includes extensions towards incorporating controllable loads [27] to address a social-welfare optimization problem [14].

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